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Banach Contraction Principle, D-Metric Space & G-Metric Space

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Abstract*: Banach Contraction principle is a fundamental result in metric fixed point theory. It is very popular and power full tool in solving the existence problems in pure and applied sciences. In this paper Banach contraction principle and various application of this famous principle, including G metric space, D-metric space. A huge Literature of fixed point theory in this space has developed that is impossible to summarize in this paper.*

Keywords: Banach contraction principle, G- metric space, D-metric space

1. Introduction

Fixed point has been developed for more than a Century and has become an important branch in Mathematics. It has widely applied to man y branches in pure and applied Mathematics, such as deferential equations; Integral equation, Game theory. Now a Day Fixed point theorem playing a vital role to various metric spaces.
Fixed point theorems are example of existence theorems in the Science that they assert the existence of Objects such as Solutions to functional equation, but not necessarily, methods for finding such solution.
It is well known that metric fixed point theory provides essential tools for solving problems arising in various branches of mathematics and other Science. Many people over the past Seventy years have tried to Generalize the definition of metric space and Corresponding fixed point theory. This trend still continues.

2. Banach fixed point Theorem (Contraction mapping theorem)

Statement:- Every contraction mapping T on a complete metric space (X, D)has a unique fixed point.

Proof:- Let α be a positive real number such that $0 < \alpha < 1$ and $d(Tx, Ty) \leq d(x, y) \quad \forall\, x, y \in X$

Let x_0 be any arbitrary fixed point.

Step -I $< x_n > \;= < x_1, x_2, - - - - - - >$

$T: X \to X, Tx \in X$

$x_1 = T\, x_0, \quad x_0 \in X$

$x_2 = T\, x_1$

$x_2 = T(T\, x_0) \quad = T^2\, x_0$

$x_n \quad = T^n\, x_0$

Where $< x_n >$ x_n ‘ are called iterates of x.

If $n > m$

$d(x_n, x_m) = d(T^m\, x_0, T^n\, x_0)$

$\leq \alpha\, d((T^{m-1}\, x_0, T^{n-1}\, x_0)$

$d(x_n, x_m) \leq \alpha^m\, d((x_0, T^{n-m}\, x_0)$

$d(x_n, x_m) \leq \alpha^m\, d((x_0, x_{n-m})$ --------(1)

Consider $d((x_0, x_{n-m}\quad) \leq d((x_0, x_1) + d((x_1, x_2) + d((x_2, x_3) +$ ----- $d((x_{n-m-1}, x_{n-m})$ --------(2)

Now $d((x_1, x_2) = d(Tx_0, T\, x_1) \leq \alpha\, d((x_0, x_1)$

$d((x_2, x_3) = d(Tx_1, T\, x_2) \leq \alpha\, d((x_1, x_2)$

$\leq \alpha^2\, d((x_0, x_1)$

$d((x_{n-m-1}, x_{n-m}) \leq \alpha^{n-m-1}\, d((x_0, x_1)$

Substituting in equation (2)

$d((x_0, x_{n-m}) = \quad d((x_0, x_1)\ \{1 + \alpha + \alpha^2 + - - - - - + \alpha^{n-m-1}\}$

$= d((x_0, x_1)\, (\frac{1-\alpha^{n+m}}{1-\alpha})$

$= \frac{d((x_0, x_1)}{1-\alpha} - \alpha^{n+m}\ \frac{d((x_0, x_1)}{1-\alpha})$

$d((x_0, x_{n-m}) \leq \frac{d((x_0, x_1)}{1-\alpha}$ -------------------(3)

Use (3) Rewrite equation (1)

$d(x_n, x_m) = d(x_0, x_1)\, \frac{\alpha^m}{1-\alpha}$ -----------------------(4)

$0 < \alpha < 1$ & $d((x_0, x_1) \geq 0$

$x_0 = x_1$

$x_0 = Tx_0$

x_0 is fixed point

Therefore $\varepsilon > 0$ we can find a Positive number such that

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$$\alpha^m \frac{d((x_0,\ x_1)}{1-\alpha} < \varepsilon, \forall n \geq p$$

With the choice of p ,we find that $d((x_m,\ x_n) < \varepsilon\ \ \forall m,n \geq P$

Therefore $< x_n >$ is a Cauchy sequence .

Since(X,D) is a complete metric spaces

Therefore $< x_n >$ Converges $\rightarrow x$

i.e $x_n \rightarrow x$

Step-II

We will x is fixed point of T.

Let $d(x,T) \leq\ d((x,\ x_n) + d((x_{n,}Tx)$

$\leq\ d((x,\ x_n) + d((Tx_{n,-1},Tx)$

$d(x,Tx) \leq\ d((x,\ x_n) +\ \alpha\ d((x_{n-1}\ ,x)$ ------------(5)

$$d(x,\ x_n) < \frac{\varepsilon_0}{2}\quad \forall\ \ n \geq q$$

$$d(\ x_n\ ,x) < \frac{\varepsilon_0}{2\alpha}\quad \forall\ \ n \geq r$$

$r = Max < a,r >$

$$d(x,\ x_n) < \frac{\varepsilon_0}{2}\quad \forall\, n \geq r$$

$$d(\ x_{n-1}\ ,x) < \frac{\varepsilon_0}{2\alpha}\quad \forall\ \ n \geq r$$

Now put in (5)

$d(x,Tx) \leq \frac{\varepsilon_0}{2} + \alpha \frac{\varepsilon_0}{2\alpha}\qquad \forall\ \ n \geq$

$d(x,Tx)\ \leq\ \ \varepsilon_0\quad \forall\ \ n \geq r$

Since $d(x,Tx)$ is non-negative real number which is smaller than every positive real number.

$d(x,\mathrm{T}x_0)\ \ =\ \ \ 0$

$Tx = x$

x is a fixed point of T

We shall now show that T has no fixed Point other than x.

Let x & y be two fixed point of T.

$Tx = x,Ty = y$,

$d(x,y) = d(Tx,Ty)$

$=\ \ \alpha\ d(Tx,Ty)$

$d(x,y) -\ \ \alpha\ d(Tx,Ty) \leq 0$

(1-α)$d(x,y) \leq 0,0 < \alpha < 1$

$1 - \alpha > 0$

Since any metric $d \geq 0$

$d(x,y) \geq 0$

Therefore $d(x,y) = 0$

$x = y$

Therefore x is the only fixed point of T .The contraction T defined on a complete metric space (X,D) has a unique fixed point.

Defination(Dhage1994)- A real funcation D on $X \times X \times X$ is said to be a D-metric on X if

D_1: $D(X,Y,Z) \geq 0$ for all $X,Y,Z \in X$ (Non- negative)

D_2: $D(X,Y,Z) = 0$ if and only if $=,Y = Z$ (Conincidence)

D_3: $D(X,Y,Z) =$ for all $D(P(X,Y,Z))$ for every $X,Y,Z \in X$ and for any permutation $P(X,Y,Z))$ of X,Y,Z(Symmetry).

D_4: $D(X,Y,Z) \leq\ D(X,Y,U) +\ D(X,U,Z) +\ D(U,Y,Z)$ for every $X,Y,Z,U \in X$(Tetrahedral inequality).

A D-metric space is a pair (X,D) where D is a D-metric on X .

Theorem: Let (X,D) be a D-metric space satisfying $D(X,Y,Y) \leq D(X,Y,Z)$ & $D(X,Y,Z) \leq\ \ D(X,U,V) +\ \ D(U,Y,V) + D(U,V,Z)$ then a real function d on $X \times X$ defined by $D(X,Y) = D(X,Y,Y)$ is a metric on X and the following are equivalent

(1) $\lim_{n\to\infty} x_n = X$ in (X,D)

(2) $\lim_{n\to\infty} x_n = X$ in (X,D)

(3) $\lim_{n\to\infty} x_n = X\ is\ strongly$ in (X,D)

It is clear that D is a metric on X.

(1) Assume $\lim_{n\to\infty} x_n = X$ in (X,D)

Let $\varepsilon > 0$ then there exist a positive integer m_0 such that $d(x,x_n) < \frac{\varepsilon}{2}\ D(y,y,x)\ \ n \geq m_0$

For any $n,m \geq m_0$

D$(x,x_n,x_m\ ) \leq\ \ D(y,x,x_m) + D(y,y,x_n)$

$= d(x,x_m) + d(y,y_n) < \varepsilon$

Thus (1) $\rightarrow$ (2)

Assume(2) $\lim_{n\to\infty} x_n = X$ in (X,D)

Let $\varepsilon > 0$ there exist a positive integer m_0 such that D$(x_n,x_m,x\ )< \varepsilon$ for all$m \geq m_0,n \geq m_0$

For $y \in X$ and $\geq m_0$, $D(y,y,x_0) \leq D(x,y,x_n)$

$\leq D(x,x,x_n)+\leq D(y,x,x)$

$= D(x,x_n,x_n) + D(y,y,x)$

This impliesthat $|D(y,y,x_n) - D(y,y,x)| \leq D(x,x_n,x_n)$

$< \varepsilon\ \forall\ \ n \geq m_0$

Hence $\{D(y,y,x_n)\}$ converges to $D(y,y,x)\ \forall\ y\ in\ X$ Thus (2) $\rightarrow$ (3), (3) $\rightarrow$ (2) is trivial

Assume (2) $\lim_{n\to\infty} x_n = X$ in (X,D)

Let $\varepsilon > 0$ there exist a positive integer m_0 such that D$(x_n,x_m,x\ )< \varepsilon$ for all$m \geq m_0,n \geq m_0$

$d(x,x_n) =$ D$(x,x,x_n) \leq$ D$(x,x_m,x_n\ ) < \varepsilon$

Hence $\lim_{n\to\infty} x_n = X$ in (X,D)

Thus (2) $\rightarrow$ (1)

Definition: Let (X,G_m) be a generalised metric space .A sequence $< x_n >$ in X is said to be G_m Cauchy if for every $\varepsilon > 0$ there exist $N \in N$ such that $G_m\ \ (\ \ x_{n1}\ \ ,\ \ x_{n2}\ ,x_{n3} \dots \dots \dots \dots \dots .. x_{n1}) < \varepsilon\ \forall n1,n2 \dots . \geq N.$

$Propostion$:- Let (X, G_m) be a generalised metric space .A sequence $< x_n >$ in X is said to be G_m Cauchy if for every $\varepsilon > 0$ there exist $N \in N$ such that $G_m(x_{n1}, x_{n2}, x_{n3} \ldots \ldots \ldots \ldots \ldots .. x_{n1}) < \varepsilon \ \forall n1, n2 \ldots . \geq N.$
$Proof$: If $< x_n >$ is an G_m Cauchy then the result follows from definition Let $G_r: x^r \rightarrow R^+$,$(r \geq 3)$ be a genrealised r metric the following are equivalent:
(1) The sequence $< x_n >$ is G_r $convergent\ to\ x$.
(2) $d_G(x_n, x) \rightarrow 0 as\ n \rightarrow \infty$
(3) $G_r((x_n, x_n, x_n, \ldots . x_n, x) \rightarrow 0\ as\ n \rightarrow \infty$
(4) $G_r((x_n, x \ldots \ldots . x) \rightarrow 0\ as\ n \rightarrow \infty$
Conversely suppose that the condition
$G_m(x_{n1}, x_{n2}, x_{n3} \ldots \ldots \ldots \ldots \ldots .. x_{n1}) < \varepsilon \ \forall n1, n2 \ldots . \geq N.$
holds for a sequence $< x_n >$ in X then for $n1, n2 \ldots . \geq N.$
We have $G_m(x_{n1}, x_{n2}, \ldots \ldots \ldots x_{n3}\ ) \leq$
$G_m(x_{n1}, x_{n2}, \ldots \ldots \ldots x_{n3}) + G_m(x_{n3}, x_{n2}, \ldots \ldots \ldots x_{n3})$
$< \varepsilon + \varepsilon$
$= 2\varepsilon$
Continuing the above argument for $n1, n2 \ldots . \geq N.$
We have $G_m(x_{n1}, x_{n2}, \ldots \ldots \ldots x_{nm}) < (m-1)\varepsilon$
i.e $< x_n >$ is G_m – Cauchy.

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