International Journal of Engineering Applied Science and Management

ISSN (Online): 2582-6948
Vol. 1 Issue 1, June 2020

Fixed Point Theorem in M- Fuzzy Metric Space

Dr. Varun Singh
Assistant Professor, TIT, Bhopal
varunsinghhh@gmail.com

Abstract

In this paper we prove Banach type fixed point theorem for complete M – fuzzy metric space. Also we establish some fixed point theorems for generalized contraction mappings in M – fuzzy metric spaces.

Keywords: M – Fuzzy metric space, M – Fuzzy contraction mapping, Fixed point theorem.

1. Introduction

In 1965, Zadeh introduced the famous theory of fuzzy sets and used it as a tool for dealing with uncertainty arising outof lack of information about certain complex system. Fixed point theorems in fuzzy mathematics are emerging with vigorous hope and vital trust. It appears that Kramosil and Michalek's study of fuzzy metric spaces paves a way forvery soothing machinery to develop fixed point theorems for contractive type maps.

Recently Sedghi and Shope [7]introduced D* - metric space as a probable modification of the definition of D - metric introduced by Dhage, and provesome basic properties in D* - metric spaces. Using D* - metric concepts, they [7] define M – fuzzy metric space and proved a common fixed point theorem in it. In this paper we prove Banach type fixed point theorem for complete M –fuzzy metric space. Also we prove some fixed point theorems for generalized contraction mappings in M – fuzzy metric space.

2. Preliminaries

Definition: 2.1:- Let X be a nonempty set. A generalized metric (or D* - metric) on X is a function: $D^* : X^3 \rightarrow [0,\infty)$ thatsatisfies the following conditions for each $x, y, z, a \in X$

(i) $D^*(x, y, z) \geq 0$

(ii) $D^*(x, y, z) = 0,\ iff\ \ x = y = z$

(iii) $D^*(x, y, z) = D^*(p\{x, y, z\})$ (symmetry) where p is a permutation function,

(iv)$D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$

The pair (X, D*), is called a generalized metric (or D* - metric) space.

Immediate examples of D* - metric are

(a) D* (x, y, z) = max {d(x, y), d(y, z), d(z, x) },

(b) D* (x, y, z) = d(x, y) + d(y, z) + d(z, x).

Here, d is the ordinary metric on X.

Definition: 2.2:- A fuzzy set M in an arbitrary set X is a function with domain X and values in [0, 1].

Definition: 2.3:- A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if it satisfies the followingconditions

(i) * is associative and commutative,

(ii) * is continuous,

(iii) $a * 1 = a$for all a $a \in [0,1]$,

(iv) $a * b \leq c * d$whenever$a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples for continuous t-norm are $a * b = ab$ and $a * b = min\{a, b\}$.

Definition:2.4:-The 3-tuple (X, M,$*$) is called a fuzzy metric space if X is an arbitrary non empty set, $*$ is a continuous t-norm and M is a fuzzy seton $X^2 \times [0,\infty)$ satisfying the following conditions for each $x, y, z \in X$ and$t, s > 0$,

(FM-1) $M(x, y, t) > 0$,

(FM-2) $M(x, y, t) = 1$ *if and only if* $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$,

(FM-4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$

(FM-5) $M(x, y, \cdot): (0, \infty) \rightarrow [0, 1]$ *is continuous*

Definition:2.5:- A 3-tuple $(X, M,*)$ is called M – fuzzy metric space if X is an arbitrary non empty set, $*$ is a continuoust-norm, and M is a fuzzy set on $X^3 \times [0,\infty)$, satisfying the following conditions for each $x, y, z, a \in X$ and $t, s > 0$

(FM – 1) $M(x, y, z, t) > 0$

(FM – 2) $M(x, y, z, t) = 1$ *iff* $x = y = z$

(FM – 3) $M(x, y, z, t) = M(p\{x, y, z\}, t)$, where p is a permutation function

(FM – 4) $M(x, y, a, t) * M(a, z, z, s) \leq M(x, y, z, t + s)$

(FM – 5) $M(x, y, z, \cdot): (0, \infty) \rightarrow [0, 1]$ is continuous

(FM – 6) $\lim_{t\to\infty} M(x, y, z, t) = 1$.

Example: 2.6:- Let X be a nonempty set and D* is the D* - metric on X. Denote $a * b = a.b$for all $a, b \in [0, 1]$. For each$t \in (0,\infty)$, define:

$$M(x, y, z, t) = \frac{t}{t + D^*(x, y, z)}$$

for all $x, y, z \in X$, then $(X, M,*)$ is a M- fuzzy metric space. We call this M-fuzzy metric induced by D* - metric space.

Thus every D* - metric induces a M-fuzzy metric.

Lemma: 2.7:-[7] Let (X, M, *) be a M-fuzzy metric space. Then for every t > 0 and for every $x, y \in X$ we have $M(x, x, y, t) = M(x, y, y, t)$.

Lemma: 2.7:-[7] Let (X, M, *) be a M –fuzzy metric space. Then M (x, y, z, t) is non-decreasing with respect to t, for allx, y, z in X.

Definition: 2.8:- Let $(X, M,*)$ be a M-fuzzy metric space. For t > 0, the open ball $B_M(x, r, t)$ with center $x \in X$ andradius $0 < r < 1$ is defined by

$$B_M(x, r, t) = \{y \in X: M(x, y, y, t) > 1 - r\}.$$

A subset A of X is called open set if for each $x \in A$ there exist $t > 0$ and $0 < r < 1$ such that $B_M(x, r, t) \subseteq A$.

Definition: 2.9:- Let $(X, M,*)$ be a M – fuzzy metric space and $\{x_n\}$ be a sequence in X

(a) $\{x_n\}$is said to be converges to a point $x \in X$ if

$$\lim_{n\to\infty} M(x, x, x_n, t) = 1 \quad \text{for all } t > 0$$

(b)$\{x_n\}$is called Cauchy sequence if

$$\lim_{n\to\infty} M(x_{n+p}, x_{n+p}, x_n, t) = 1 \quad \text{for all } t > 0, \; p > 0$$

(c) A M – fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

3. Main result

Theorem: 3.1:- Let$(X, M,*)$ be a M-fuzzy metric space with $t * t \geq t$ for all $t \in [1,0]$ *and* $T: X \rightarrow X$ be a mapping such that for all $x, y, z \in X$ *and* $t > 0$ with $x \neq y \neq z$

ISSN (Online): 2582-6948
Vol. 1 Issue 1, June 2020

$$M(T_x,T_y,T_z,t) > min\left\{M(X,Y,Z,T),\frac{M(x,y,z,t)*M(y,z,T_y,t)}{M(x,x,T_x,t)*M(y,y,T_y,t)}\right\}$$

For any point $x_0 \in X$ the sequence$\{T^n(x_0)\}$ has a subsequnce converges to u.
Then u is a unique fixed point of T.

Proof: Let$x_0 \in X$ be any arbitrary fixed point.
Define a sequence $\{x_n\}$ in X as $x_{n+1} = T_{x_n}$ for $n = 0,1,2,\ldots$
Suppose$x_n = x_{n+1}$ for some n.
Then $x_n = T_{x_n}$ thus $x_n = u$ is a fixed point of T.

Let $x_n \neq x_{n+1}$ for all n.For n$\geq$ 1 we have.

$M(x_n,x_n,x_{n+1},t) = M(Tx_{n-1},Tx_{n-1},Tx_n,t)$

$$\geq min\left\{M(x_{n-1},x_{n-1},x_n\ ,\ t),\frac{M(x_{n-1},x_{n-1},T_{x_{n-1}},t)*M(x_{n-1},x_n,T_{x_{n-1}},t)}{M(x_{n-1},x_{n-1},T_{x_{n-1}},t)*M(x_{n-1},x_{n-1},T_{x_{n-1}},t)}\right\}$$

$$= min\left\{M(x_{n-1},x_{n-1},x_n\ ,\ t),\frac{M(x_{n-1},x_{n-1},x_n,t)*M(x_{n-1},x_n,x_n t)}{M(x_{n-1},x_{n-1},x_n,t)*M(x_{n-1},x_{n-1},x_n,t)}\right\}$$

$$= min\left\{M(x_{n-1},x_{n-1},x_n\ ,\ t),\frac{M(x_{n-1},x_{n-1},x_n,t)}{M(x_{n-1},x_{n-1},x_n,t)}\right\}$$

$$= min\{M(x_{n-1},x_{n-1},x_n\ ,\ t),1\}$$

Therefore,

$$M(x_n,x_n,x_{n+1}\ ,\ t) \geq M(x_{n-1},x_{n-1},x_n\ ,\ t)$$

Thus, $\{M(x_n,x_n,x_{n+1},t)\}$ is monotonically sequence of positive real number bounded above by 1.

It is convergent to a positive real number, say L

Therefore $\lim_{n\to\infty} M(x_n,x_n,x_{n+1},t) = L$
Also the sequence $\{M(x_n,x_n,x_{n+1},t)\}$ has a subsequence$M(x_{n_k},x_{n_k},x_{n_{k+1}},t) = L$

Now we prove that L=1
Suppose L< 1

Since $x_n = T^n x_0$ has a subsequence x_{n_k} converges to u.
We have$\lim_{k\to\infty} M(x_{n_k},x_{n_k},u,t) = 1$ ……………….. (i)
Since $1 > L = \lim_{k\to\infty} M(x_{n_k},x_{n_k},x_{n_{k+1}},t)$

$$1 > L \geq \lim_{k\to\infty} M\left(x_{n_k},x_{n_k},u,\frac{t}{2}\right) * M\left(u,x_{n_{k+1}},x_{n_{k+1}},\frac{t}{2}\right) = 1*1$$

$1 > 1*1$
Which is contradiction, Hence L=1
Therefore $M(x_{n_k},x_{n_k},x_{n_{k+1}},t) = 1$
and$\lim_{n\to\infty} M(x_n,x_n,x_{n+1},t) = 1$
Thus$\{x_n\}$ is cauchy sequence.

Suppose $L < 1$
Since $x_n = T^n{}_{x_0}$ has a subsequence x_{n_k}converse to u.
We have $\lim_{k\to\infty} M(x_{n_k},x_{n_k},u,t) = 1$
Now $1 > L = M(x_{n_k},x_{n_k},x_{n_{k+1}},t) = 1$

$$\geq \lim_{k\to\infty}\left\{M\left(x_{n_k},x_{n_k},u,\frac{t}{2}\right) * M\left(u,x_{n_{k+1}},x_{n_{k+1}},\frac{t}{2}\right)\right\}$$
$$=1*1$$

$1> 1*1$

Which is contradiction.

Now we prove u is fixed point of T.
Suppose $u \neq T(u)$ we have.

$$M(u,u,T_u t) = \lim_{k\to\infty} M(x_{n_{k+1}},x_{n_{k+1}},T_u,t) = \lim_{k\to\infty} M(x_{n_k},x_{n_k},T_u,t)$$

$$> \lim_{k\to\infty} min\left\{M(x_{n_k},x_{n_k},u,t),\frac{M\left(x_{n_k},x_{n_k},T_{x_{n_k}},t\right) * M(x_{n_k},u,T_{n_k},t)}{M\left(x_{n_k},x_{n_k},T_{x_{n_k}},t\right) * M(x_{n_k},x_{n_k},T_{n_k},t)}\right\}$$

$$= \lim_{k\to\infty} min\left\{M(x_{n_k},x_{n_k},u,t),\frac{M(x_{n_k},x_{n_k},T_{x_{n_k}},t)*M(x_{n_k},u,T_{n_k},t)}{M(x_{n_k},x_{n_k},T_{x_{n_k}},t)*M(x_{n_k},x_{n_k},T_{n_k},t)}\right\}$$

=
$\lim_{k\to\infty} min\left\{M\left(x_{n_k}, x_{n_k}, u, t\right), \frac{M\left(x_{n_k},x_{n_k},x_{n_{k+1}},t\right)* M\left(x_{n_k},u,x_{k+1},t\right)}{M\left(x_{n_k},x_{n_k},x_{n_{k+1}},t\right)* M\left(x_{n_k},x_{n_k},x_{n_{k+1}},t\right)}\right\}$

$= \min\left\{M(u,u,u,t), \frac{M(u,u,u,t)*M(u,u,u,t)}{M(u,u,u,t)*M(u,u,u,t)}\right\}$

$=min\{M(u,u,u,t); 1\}$
$= \min\{1,1\}$
$=1$
$M(u,u,Tu,t) \geq 1 \ldots\ldots\ldots$
Hence $u = Tu$.

4. Uniqueness

Suppose there exists $\vartheta \in X$ *such that* $T_u = \vartheta$ *and* $\vartheta \neq u$

Now consider $M(u,u,\vartheta,t) = M\left(T_u, T_u, T_{\vartheta,}t\right)$

$$> \min\left\{M(u,u,\vartheta,t), \frac{M(u,u,T_u,t) * M(u,\vartheta,T_u,t)}{M(u,u,T_u,t) * M(u,u,T_u,t)}\right\}$$

$$= \min\left\{M(u,u,\vartheta,t), \frac{M(u,u,u,t) * M(u,\vartheta,u,t)}{M(u,u,u,t) * M(u,u,u,t)}\right\}$$

$$= \min\left\{M(u,u,\vartheta,t), \frac{1*M(u,\vartheta,u,t)}{1*1}\right\}$$

$= \min\{M(u,u,\vartheta,t), M(u,\vartheta,u,t)\}$

$= M(u,u,\vartheta,t)$

$M(u,u,\vartheta,t) > M(u,u,\vartheta,t)$

Which is contradiction.

Therefore u is a unique fixed point of T.

Reference

[1] Dhage .B.C, Generalised metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc., 84(4), (1992),329-336.

[2] Grabiec .M, Fixed points in fuzzy metric spaces, Fuzzy Sets and Systems, 27 (1988), 385-389.

[3] Jin-Xuan Fang, On fixed point theorems in fuzzy metric spaces, Fuzzy Sets and System, 46 (1992), 107-113.

[4] Kramosil .I and Michalek .J, Fuzzy metric and statistical metric spaces, Kybernetica, 11 (1975), 326-334.

[5] Naidu .S.V.R, Rao .K.P.R and Srinivasa .R.N, On the topology of D-metric spaces and the generation of D-metricspaces from metric space, Internet. J. Math. Math. Sci., 51 (2004), 2719-2740.

[6] Naidu .S.V.R, Rao .K.P.R and Srinivasa .R.N, On the convergent sequences and fixed point theorems in D-metricspaces, Internet. J. Math. Math. Sci., 12 (2005), 1969-1988.

[7] Sedghi .S and Shobe .N, Fixed point theorem in M -fuzzy metric spaces with property (E), Advances in FuzzyMathematics, Vol.1, No.1 (2006), 55-65.

[8] Schweizer .B and Sklar .A, Statistical metric spaces, Pacific. J. Math., 10 (1960), 314-334.

[9] Veerapandi .T, Jeyaraman .M and Paul Raj Josph .J, Some fixed point and coincident point theorem in generalizedM -fuzzy metric space, Int. Journal of Math. Analysis, Vol.3, (2009), 627-635.

[10] Zadeh .L. A, Fuzzy sets, Information and Control, 8 (1965), 338-353.